International Journal of Theoretical Physics, Vol. 2, No. 2 (1969), pp. 161–187.

Relativistic Fluid Dynamics In A Non-Vacuum Régime†

M. Z. v. KRZYWOBLOCKI

Michigan State University

Abstract

The author obtains the fundamental aspects of relativistic fluid dynamics in a non-vacuum régime. As the basic model is taken the special theory of relativity in the form proposed by Einstein (1907), Fock (1955) and others. The model takes account of the influence of the gravitational field upon the velocity of the propagation of light.

Introduction

The relativistic theory of fluid dynamics was constructed in the past by a few writers, notably by Taub (1948) and others. The characteristic feature of those formulations is that they are based upon the special theory of relativity with the reference velocity equal to the velocity of the propagation of light *in vacuo*. But, in practice, the light in the interplanetary or even interstellar régime is subject to the action of the gravitational field and consequently the velocity of its propagation is smaller than that *in vacuo*. If the light is used as the tested and measured signal, some corrections should be introduced to take account of the influence of the gravitational field. The model of the special theory of relativity with the action of the gravitational field taken into account, as proposed by Einstein (1907), Fock (1955) and others, is the basic model in the present work. In the first sections there is derived the theory of relativistic fluid dynamics not in a vacuum, based primarily upon the model of Taub. The use of the constant velocity of light in a medium and of the local (contact) spatial coordinate system enables one to present the final expressions for the motion of a fluid and for the normal shock relations in a form directly reducible to Taub's equations. In the next section there are derived equations of motion and shock relations in the Riemannian space to demonstrate what kind of simplifications have to be introduced to reduce the formalism to that in the (local) Euclidean space. The equations in the Riemannian space were not treated, due to the extreme difficulties. A hypothetical example, demonstrating the influence of the action of the gravitational field upon the normal shock relations in the relativistic fluid dynamics not in a vacuum, is dealt with at the end of the work.

† The research was sponsored by the Research Department, Michigan State University, under the directorship of Mr. John Hoffman, and this is gratefully acknowledged.

1. *Fundamental Aspects of the Relativistic Models*

1.1. *Relativistic Models*

In 1905 Einstein proposed his first model of the special theory of relativity involving the velocity of the propagation of light *in vacuo* (Einstein, 1905). This model, as is well known, is an idealistic model since *in vacuo* nothing, not even a light, can exist. Two years later (Einstein, 1907) he proposed his second relativistic model involving the effect of the gravitational field and derived the formula for the metric of the space-time in the form:

$$dx^2 + dy^2 + dz^2 = [c(1 + \phi c^{-2})]^2 \, dt^2 \tag{1.1.1}$$

ϕ being the gravitational potential. Bateman (1910) derived the general infinitesimal spherical wave transformation equations of the form:

$$x' = x + \varepsilon[p(x^2 - y^2 - z^2 + t^2) + 2qxy + 2rxz + 2sxt + \mu x + hy + gz + lt + a] \tag{1.1.2}$$

where all the coefficients are constant, $\varepsilon \ll 1$, and terms containing ε^2 are neglected. After some manipulations, with $\gamma = -2\varepsilon$ denoting a constant acceleration of a point moving along a straight line, Bateman obtains equations:

$$x' = x - \tfrac{1}{2}\gamma(y^2 + z^2 - x^2 - t^2); \qquad y' = y(1 + \gamma x) \tag{1.1.3}$$

$$z' = z(1 + \gamma x); \qquad t' = t(1 + \gamma x) \tag{1.1.4}$$

which agree with those obtained by Einstein in 1907. One should keep in mind that Bateman does not require that the velocity of the propagation of signals refers to the velocity of the propagation of light *in vacuo*, but to the propagation in any homogenous or heterogenous medium. But, analogously to Einstein and Taub, Bateman does not mention that the velocity of light in media, mentioned above, is necessarily equal to the velocity of light *in vacuo*.

One may mention Whitehead's intuitive theory involving mass impetus, electromagnetic impetus and the total impetus. The critical velocity does not refer to the velocity of light but merely expresses the fact that a lapse of time and a stretch of spatial route can be congruent to each other. From the formalism of Lorentz transformations it is evident that they are valid for any arbitrary constant reference velocity different from the velocity of the propagation of light *in vacuo*. In 1951 Rosen (1952) put forward his idea of c'-relativities, where any c' = constant may be chosen as a reference velocity in Lorentz transformations. Fock (1955) proposed the following forms of the four-dimensional space-time metric:

$$ds^2 = (c^2 - 2U)\, dt^2 - (dx^2 + dy^2 + dz^2) \tag{1.1.5}$$

or

$$ds^2 = (c^2 - 2U)\, dt^2 - (1 - 2Uc^{-2})(dx^2 + dy^2 + dz^2) \tag{1.1.6}$$

with $U \equiv \phi$. In 1960 the author† applied a space-time metric of the form

$$ds^2 = (dx^2 + dy^2 + dz^2) - c^2(x, y, z)\,dt^2 \tag{1.1.7}$$

with $c <$ velocity of light *in vacuo*, to investigate approximate transformation formulas between $\{x\}$ and $\{x'\}$, and the mass-energy relation. The latter one is uneffected by the fact that c is a position function. In a series of his papers† in the years 1961–65 the author developed the approach to the special theory starting from the energy equation of a nonviscous fluid flow in classical Newtonian mechanics (Bernoulli's equation), resulting in the following form for the space-time metric:

$$ds^2 = (dx^2 + dy^2 + dz^2) - I^2\,dt^2 \tag{1.1.8}$$

I^2 stands for $I^2 = 2(\frac{1}{2}c^2 \mp$ potential energy injected into or ejected from the system, work of the external forces, etc.).‡ This may be considered to be the most general form of the space-time metric in the remodelled special theory of relativity. There are physicists who oppose calling the above model of the theory of relativity by the name of the special theory of relativity.§ We shall use the nomenclature: the classical and the remodelled special theory of relativity. In the present work we assume a metric for the four-dimensional space-time $\{\bar{x}\}$ in the form:

$$-ds^2 = \bar{a}_{jk}\,d\bar{x}^j\,d\bar{x}^k - c^{-2}I^2(d\bar{x}^4)^2 = \bar{a}_{\sigma\rho}\,d\bar{x}^\sigma\,d\bar{x}^\rho \tag{1.1.9}$$

$$\bar{a}_{11} = \bar{a}_{22} = \bar{a}_{33} = 1; \qquad \bar{a}_{44} = -c^{-2}I^2; \qquad \bar{a}_{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \tag{1.1.10}$$

and $\bar{x}^4 = ct$. Throughout the work the Latin indexes take up the values 1, 2, 3, and the Greek indexes the values 1 to 4. The contravariant components of the metric tensor, $\bar{a}^{\sigma\rho}$, in $\bar{a}^{\sigma\rho}\,d\bar{x}_\sigma d\bar{x}_\rho$, are:

$$\bar{a}^{11} = \bar{a}^{22} = \bar{a}^{33} = 1; \qquad \bar{a}^{44} = -c^2I^{-2}; \qquad \bar{a}^{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \tag{1.1.11}$$

From (1.1.9) we obtain a formula which will be used below:

$$dt/ds = [I(1 - \bar{q}^2 I^{-2})^{1/2}]^{-1}; \qquad \bar{q}^j = d\bar{x}^j/dt; \qquad \bar{q}^2 = \bar{a}_{jk}\bar{q}^j\bar{q}^k \tag{1.1.12}$$

There is more than one reason for the necessity of the derivation of the relativistic formulas in various fields like hydrodynamics when the reference velocity of signals is less than the velocity of light *in vacuo*. Firstly, the velocity of light is not really a constant. It depends upon the intensity of the gravitational field, as Einstein had demonstrated. Second, the velocity of

† Krzywoblocki, v. M. Z. 1. On the General Form of the Special Theory of Relativity. I, II, III, IV. *Acta physica austriaca*, Vol. 13, No. 4, 387–394 (1960); Vol. 14, No. 1, 22–28 and 39–49 (1961); Vol. 14, No. 2, 239–241 (1961).

2. Special Relativity—A Particular Energy Formulation in Newtonian Mechanics? I, II. *Acta physica austriaca*, Vol. 15, No. 3, pp. 201–212 and 251–261 (1962).

3. On the Fundamentals of the Relativistic Theories. *Acta physica austriaca*, Vol. 15, No. 4, 320–336 (1962).

‡ Various models of the special theory of relativity are constructed by Møller (1952).

§ Bondi (Trautmann *et al.*, 1964) states that: 'It is occasionally asserted that as soon as we have acceleration then we cannot work with special relativity. This is quite untrue,...'.

the propagation of the light in a medium is given by $v = cn^{-1}$, where c is the velocity of light *in vacuo*, n is the ratio of the velocity of light in empty space to the velocity, v, in the medium in question and n is called the index of refraction. Some works on physics give the value of n for electromagnetic waves as $(\varepsilon/\mu)^{1/2}$, where ε is the dielectric coefficient (not necessarily a constant) and μ is the magnetic permeability.

The velocity of propagation of electromagnetic waves depends on several phenomena, and parameters like diffraction, refraction, dispersion, variable index of refraction, variable dielectric coefficient, and so on.

1.2. *Relativistic Mechanics of a Mass Point*

Using (1.1.9) we obtain the contravariant four-velocity vector and its covariant and contravariant components:

$$\bar{z}^\lambda = d\bar{x}^\lambda/ds; \qquad \bar{z}^j = \bar{q}^j\beta^{-1}; \qquad \beta = [I(1-\bar{q}^2 I^{-2})^{1/2}] \tag{1.2.1}$$

$$\bar{z}^4 = \bar{q}^4\beta^{-1}; \qquad \bar{q}^4 = d\bar{x}^4/dt = c; \qquad \bar{z}_\sigma = \bar{a}_{\sigma\rho}\bar{z}^\rho \tag{1.2.2}$$

The magnitude of the four-velocity vector is:

$$\bar{a}_{\sigma\rho}\bar{z}^\sigma\bar{z}^\rho = -1 \qquad \text{and} \qquad \bar{z}\,.\,\delta\bar{z}_\sigma/\delta s = 0 \tag{1.2.3}$$

since the absolute derivative of the metric tensor is equal to zero. The symbol $\delta/\delta s$ denotes the absolute derivative. Let us introduce the Lagrangian, $\mathscr{L}$, and calculate the four-momentum vector:

$$\mathscr{L} = \tfrac{1}{2}m_0 c^2 \bar{a}_{\sigma\rho}\bar{z}^\sigma\bar{z}^\rho; \qquad \bar{P}_\sigma = \partial\mathscr{L}/\partial\bar{z}^\sigma = m_0 c^2 \bar{a}_{\sigma\rho}\bar{z}^\rho \tag{1.2.4}$$

where m_0 denotes the mass at rest. Using (1.1.10) and (1.2.1) in (1.2.4) gives:

$$\bar{P}_j = cm_0(c^{-1}\beta)^{-1}\bar{q}^j; \qquad \bar{P}_4 = -m_0(c^{-1}\beta)^{-1}I^2 \tag{1.2.5}$$

With $\bar{M}$ denoting the inertial relativistic mass we get:

$$\bar{M} = m_0(c^{-1}\beta)^{-1}; \qquad \bar{P}_j = c\bar{M}\bar{q}_j; \qquad \bar{P}_4 = -\bar{M}I^2 \tag{1.2.6}$$

$$\bar{P}^j = \bar{a}^{j\sigma}\bar{P}_\sigma = m_0 c^2 \bar{z}^j = c\bar{M}\bar{q}^j \tag{1.2.7}$$

$$\bar{P}^4 = \bar{a}^{4\sigma}\bar{P}_\sigma = \bar{a}^{44}\bar{P}_4 = \bar{M}c^2 \tag{1.2.8}$$

The standard tensor calculus gives the formulae for the absolute derivatives:

$$\delta T^\sigma/\delta s = dT^\sigma/ds + \begin{Bmatrix}\sigma\\ \mu\nu\end{Bmatrix} T^\mu\, dx^\nu/ds, \text{ etc.} \tag{1.2.9}$$

which lead to the contravariant four-force vector components:

$$F^\sigma(\bar{x}) = \delta\bar{P}/\delta s; \qquad F^j(\bar{x}) = [d\bar{P}^j/dt + \tfrac{1}{2}\bar{a}^{jk}(c^{-2}I^2)_{,k}\bar{P}^4 c]\, dt/ds \tag{1.2.10}$$

$$F^j(\bar{x}) = (c^{-1}\beta)^{-1}\bar{F}^j(\bar{x}); \qquad \bar{F}^j(\bar{x}) = d/dt(\bar{M}\bar{q}^j) + \tfrac{1}{2}\bar{a}^{jk}\bar{M}(I^2)_{,k}$$
$$= \delta/\delta t(\bar{M}\bar{q}^j); \tag{1.2.11}$$

and the covariant components:

$$F_j(\bar{x}) = \bar{a}_{j\sigma} F^\sigma(\bar{x}) = (c^{-1}\beta)^{-1} \bar{F}_j(\bar{x}); \qquad \bar{F}_j(\bar{x}) = \delta/\delta t(\bar{M}\bar{q}_j) \quad (1.2.12)$$

$$F^4(\bar{x}) = \bar{a}^{4\sigma} F_\sigma(\bar{x}) = \beta^{-1} c^2 I^{-2} d/dt(\bar{M}I^2) = \beta^{-1} \bar{F}^4(\bar{x}) \quad (1.2.13)$$

$$F_4(\bar{x}) = \beta^{-1} \bar{F}_4(\bar{x}) = -\beta^{-1}[d/dt(\bar{M}I^2)] \quad (1.2.14)$$

From $(1.2.3)_2$, $(1.2.4)_2$ and $(1.2.10)_1$, and then using $(1.2.1)_2$, (1.2.11) and (1.2.14), we obtain:

$$\bar{z}^\sigma F_\sigma(\bar{x}) = 0 \qquad \text{or} \qquad \bar{q}^j \bar{F}_j(\bar{x}) = d/dt(\bar{M}I^2) \quad (1.2.15)$$

Defining the real energy of a particle by $\bar{\varepsilon}^*$, we get using $(1.2.6)_3$:

$$\bar{\varepsilon}^* = \bar{M}I^2; \qquad \bar{P}_4 = -\bar{\varepsilon}^* \quad (1.2.16)$$

Defining the total relative energy by $\bar{\varepsilon}_t^*$, using (1.2.8) and $(1.2.16)_2$, we obtain the maximum energy a particle may possess:

$$\bar{\varepsilon}_t^* = \bar{P}^4 = -c^2 I^{-2} \bar{P}^4 = c^2 I^{-2} \bar{\varepsilon}^* = \bar{M}c^2 \quad (1.2.17)$$

1.3. *Local Coordinates*

To avoid the operations in the Riemannian space-time we propose the local coordinates with the metric of the corresponding space-time of the form:

$$-(ds)^2 = \bar{g}_{\sigma\rho}\, d\bar{y}^\sigma\, d\bar{y}^\rho = \bar{g}_{jk}\, d\bar{y}^j\, d\bar{y}^k - I_0{}^2\, dt^2 \quad (1.3.1)$$

where the subscript zero denotes the value of I at a certain particular point '0'. When I_0 is constant in a relatively small but a finite domain around the point '0', the coefficients in (1.3.1) are constant, since $\bar{g}_{jk} = 1$ for $j = k$, $\bar{g}_{jk} = 0$ for $j \neq k$. The four-velocity vector has the components:

$$\bar{\xi}^\lambda = d\bar{y}^\lambda/ds = (d\bar{y}^\lambda/dt)(dt/ds); \qquad \bar{\xi}^j = \bar{v}^j \beta_1^{-1} \quad (1.3.2)$$

$$\beta_1 = [I_0(1 - \bar{v}^2 I_0^{-2})^{1/2}]; \qquad \bar{\xi}^4 = (c^{-1}\beta_1)^{-1} \quad (1.3.3)$$

$$dt/ds = \beta_1; \qquad \bar{v}^j = d\bar{y}^j/dt; \qquad (\bar{v})^2 = \bar{g}_{jk}\bar{v}^j \bar{v}^k \quad (1.3.4)$$

All the formulas in Section 1.2 in the $\{\bar{x}\}$-space are transformable directly into the corresponding formulae in the $\{\bar{y}\}$-space, when the following rules and transformations are preserved:

$$\bar{M} \to \bar{m} = m_0(c^{-1}\beta_1)^{-1}; \qquad \bar{q}^j \to \bar{v}^j; \qquad I \to I_0; \qquad \bar{P}_j \to \bar{p}_j \quad (1.3.5)$$

$$\bar{F}_j(\bar{x}) \to \bar{F}_j(\bar{y}); \qquad \bar{P}^4 \to \bar{p}^4 = \bar{m}c^2, \text{ etc.} \quad (1.3.6)$$

1.4. *Second Form of the Four-Dimensional Space-Time*

Let us describe a 'world point' in terms of three space coordinates $\{x^j\}$ and time $t \equiv x^4$. The corresponding metric is:

$$-(d\tau)^2 = a_{\sigma\rho}\, dx^\sigma\, dx^\rho = a_{jk}\, dx^j\, dx^k - c^{-2} I^2 (dx^4)^2; \qquad d\tau = c^{-1} ds \quad (1.4.1)$$

$$a_{jk} = c^{-2} \text{ for } j = k; \qquad a_{44} = -c^{-2} I^2; \qquad a_{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \quad (1.4.2)$$

Four-velocity:

$$z^\lambda = dx^\lambda/d\tau; \quad z^j = q^j(c^{-1}\beta_2)^{-1}; \quad \beta_2 = [I(1-q^2I^{-2})^{1/2}] \quad (1.4.3)$$

$$q^j = dx^j/dt; \quad z^4 = (c^{-1}\beta_2)^{-1} \quad (1.4.4)$$

Four-momentum:

$$P_j = m_0 c^2 z_j = c^2 Mq_j; \quad P_4 = -MI^2; \quad M = m_0(c^{-1}\beta_2)^{-1} \quad (1.4.5)$$

$$P^j = m_0 c^2 z^j = c^2 Mq^j; \quad P^4 = Mc^2 = \varepsilon_t^* \quad (1.4.6)$$

Four-force:

$$F_\sigma(x) = \delta P_\sigma/\delta\tau = m_0 c^2 \delta z_\sigma/\delta\tau; \quad F_j(x) = (c^{-1}\beta_2)^{-1}\bar{F}_j(x) \quad (1.4.7)$$

$$\bar{F}_j(x) = c^2\delta/\delta t(Mq_j) = c^2[d/dt(Mq_j) + \tfrac{1}{2}M(I^2)_{,j}] \quad (1.4.8)$$

$$F_4(x) = (c^{-1}\beta_2)^{-1}\bar{F}_4(x); \quad \bar{F}_4(x) = -d/dt(MI^2) \quad (1.4.9)$$

Energy:

$$q^j\bar{F}_j(x) = d/dt(MI^2) = d\varepsilon^*/dt \quad (1.4.10)$$

$$q_j\bar{F}^j(x) = c^{-2}I^2\bar{F}^4(x); \quad \bar{F}^4(x) = c^2I^{-2}d/dt(MI^2) \quad (1.4.11)$$

The following expression is required later.

$$\delta z^j/\delta t = dz^j/dt + \tfrac{1}{2}m_0^{-1}Ma^{kj}(c^{-2}I^2)_{,k} \quad (1.4.12)$$

Similar formulae in the $\{y\}$-space-time are:

$$-d\tau^2 = g_{\sigma\rho}dy^\sigma dy^\rho; \quad dy^4 = dt; \quad g_{11} = g_{22} = g_{33} = c^{-2};$$

$$g_{44} = -c^{-2}I_0^2 \quad (1.4.13)$$

with $g_{\sigma\rho} = 0$ for $\sigma \neq \rho$. The four-velocity:

$$\xi^\lambda = dy^\lambda/d\tau; \quad \xi^j = v^j(c^{-1}\beta_3)^{-1}; \quad v^j = dy^j/dt \quad (1.4.14)$$

$$\beta_3 = [I_0(1-v^2I_0^{-2})^{1/2}]; \quad \xi^4 = (c^{-1}\beta_3)^{-1} \quad (1.4.15)$$

The four-momentum:

$$p_j = m_0 c^2\xi_j = c^2 mv_j; \quad p_4 = -mI_0^2 = -\varepsilon^*(y) \quad (1.4.16)$$

$$p^4 = mc^2 = \varepsilon_t^*(y); \quad m = m_0(c^{-1}\beta_3)^{-1} \quad (1.4.17)$$

The four-force:

$$F_\sigma(y) = dp_\sigma/d\tau = m_0c^2d\xi_\sigma/d\tau; \quad F_j(y) = (c^{-1}\beta_3)^{-1}\bar{F}_j(y) \quad (1.4.18)$$

$$\bar{F}_j(y) = c^2d/dt(mv_j); \quad F_4(y) = (c^{-1}\beta_3)^{-1}\bar{F}_4(y); \quad \bar{F}_4 = -d/dt(mI_0^2) \quad (1.4.19)$$

The energy:

$$v^j\bar{F}_j(y) = d/dt(mI_0^2) = d\varepsilon^*(y)/dt \quad (1.4.20)$$

$$v_j\bar{F}^j(y) = c^{-2}I^2\bar{F}^4(y); \quad \bar{F}^4(y) = c^2I_0^{-2}d/dt(mI_0^2) \quad (1.4.21)$$

1.5. *Transformation of Coordinates*

Let us introduce transformation of coordinates $x \leftrightarrow X, y \leftrightarrow Y$:

$$X^j = c^{-1} x^j; \qquad X^4 = x^4 = t; \qquad d\tau^2 = -A_{\sigma\rho}\, dX^\sigma\, dX^\rho \tag{1.5.1}$$

$$A_{jk} = 1 \text{ for } j = k; \qquad A_{44} = -c^{-2} I^2; \qquad A_{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \tag{1.5.2}$$

The velocity and the force vectors are:

$$V^j(X) = dX^j/d\tau = c^{-1} z^j; \qquad V^4 = dX^4/d\tau = cI^{-1}(1 + z^2 c^{-2})^{1/2} \tag{1.5.3}$$

$$V_j(X) = cz_j; \qquad [V(X)]^2 = A_{jk}\, V^j(X)\, V^k(X); \qquad A_{\sigma\rho}\, V^\sigma\, V^\rho = -1 \tag{1.5.4}$$

$$F^j(X) = \delta/\delta\tau(m_0 c^2\, V^j(X)) = c^{-1} F^j(x) \tag{1.5.5}$$

$$\bar{F}^j(X) = c^{-1} \bar{F}^j(x); \qquad F_j(X) = cF_j(x) \tag{1.5.6}$$

In the local orthogonal Y-coordinates:

$$Y^j = c^{-1} y^j; \qquad Y^4 = y^4 = t; \qquad -d\tau^2 = G_{\sigma\rho}\, dY^\sigma\, dY^\rho - c^{-2} I_0{}^2 (dY^4)^2 \tag{1.5.7}$$

$$G_{jk} = 1 \text{ for } j = k; \qquad G_{44} = -c^{-2} I_0{}^2; \qquad G_{\sigma\rho} = 0 \text{ for } \sigma \neq \rho \quad (1.5.8$$

$$V^j(Y) = dY^j/d\tau = c^{-1} \xi^j; \qquad V^4(Y) = dY^4/d\tau = cI_0^{-1}[1 + (V(Y))^2]^{1/2} \tag{1.5.9}$$

$$V_j(Y) = c\xi_j; \qquad (V(Y))^2 = G_{jk}\, V^j(Y)\, V^k(Y); \qquad G_{\sigma\rho}\, V^\sigma\, V^\rho(Y) = -1 \tag{1.5.10}$$

$$F^j(Y) = c^{-1} F^j(y); \qquad \bar{F}^j(Y) = c^{-1} \bar{F}^j(y); \qquad F_j(Y) = cF_j(y) \tag{1.5.11}$$

2. *Relativistic Fluid Dynamics In the Non-Vacuum Régime*

2.1. *Fundamental Equations*

The relativistic Rankine-Hugoniot equations in a flat space with the reference velocity equal to the velocity of light *in vacuo* were derived by Taub (1948). In the present work we derive the relativistic hydrodynamic equations in the non-vacuum régime. In order to avoid dealing with the curvilinear Riemannian spaces, we introduce the local orthogonal, y^σ, coordinates, discussed above. The random velocity components, v^j, are measured with respect to the fixed system $\{y\}$. The motion of the fluid considered as a collection of particles of the rest mass m_0 is described in terms of:

$$\xi^j = v^j(c^{-1} \beta_3)^{-1}; \qquad v^j = c^{-1} I_0\, \xi^j (1 + c^{-2} \xi^2)^{-1/2} \tag{2.1.1}$$

$$c^{-2} \xi^2 = g_{jk}\, \xi^j\, \xi^k; \qquad v^2 c^{-2} = g_{jk}\, v^j\, v^k \tag{2.1.2}$$

with $g_{jk} = c^{-2}$ for $j = k$ and $g_{jk} = 0$ for $j \neq k$, where v^j are the components of the velocity of a particle. Let us introduce the distribution function (y^j, ξ^j, t) with the Boltzmann equation:

$$Df \equiv \frac{\partial f}{\partial t} + v^j \frac{\partial f}{\partial y^j} + \mathscr{F}^j \frac{\partial f}{\partial \xi^j} = \Delta_e f \tag{2.1.3}$$

or using (2.1.1):

$$Df = \frac{\partial f}{\partial t} + c^{-1} \xi^j (1 + \xi^2 c^{-2})^{-1/2} \frac{\partial f}{\partial y^j} + \mathscr{F}^j \frac{\partial f}{\partial \xi^j} = \Delta_e f \tag{2.1.4}$$

where $\mathscr{F}^j = m_0^{-1} c^{-2} F^j(y)$ denotes the external force per unit mass and $\Delta_e f$ is the time rate of change in f due to encounters between particles. Beginning from this point we may apply precisely the procedure used by Taub with the corresponding changes due to the fact that we operate in a non-vacuum régime.

Multiply (2.1.4) by any transport quantity $\phi(y^j, \xi^j, t)$, integrate over the entire volume of the $\{\xi^j\}$-space, apply the integration by parts with the mean values of $\phi, n.\langle\phi\rangle$, defined by:

$$n.\langle\phi\rangle = \int \phi f \, d_3 \xi; \qquad n = \int f \, d_3 \xi \tag{2.1.5}$$

and apply the notion of the summation invariants i.e., there are some functions, Ψ, characterized by conservation properties during encounters in the sense that:

$$\int \Psi^q . \Delta_e f . d_3 \xi = 0; \qquad q = 0, 1, 2, 3, 4 \tag{2.1.6}$$

$$\Psi^0 = m_0; \qquad \Psi^j = m_0 \xi^j; \qquad \Psi^4 = E = \text{total energy of a particle} \tag{2.1.7}$$

$$E = \varepsilon_t^*(y) = p^4 = mc^2 = c^2 m_0 (c I_0^{-1})(1 + \xi^2 c^{-2})^{1/2} \tag{2.1.8}$$

To obtain the law of conservation of mass we introduce the mass current four-vector:

$$U^\alpha = \int V^\alpha \, d\mu; \qquad d\mu = (1 + c^{-2} \xi^2)^{-1/2} . f . d_3 \xi \tag{2.1.9}$$

where V^α is given in (1.5.9), and insert Ψ^0 into the transformed $\int \Psi^0 . Df . d_3 \xi$, thus obtaining:

$$m_0 \, U^\alpha|_\alpha = 0 \tag{2.1.10}$$

Introducing the average velocity:

$$\bar{u}^j = n^{-1} \int v^j . f . d_3 \xi; \qquad (\bar{u})^2 = G_{jk} \bar{u}^j \bar{u}^k \tag{2.1.11}$$

and making use of (2.1.5) and (2.1.9), we obtain:

$$I_0^{-2} \beta_4^{\,2} = (1 - (\bar{u})^2 I_0^{-2}) = -n^{-2} U^\alpha U_\alpha \tag{2.1.12}$$

Define the relativistic number density n^0 as measured by an observer moving with velocity $\bar{u}^j$ with respect to the fixed coordinates, Y^j:

$$n^0 = n^2 (1 - (\bar{u})^2 I_0^{-2}) = -U^\alpha U_\alpha; \qquad \rho^0 = n^0 m_0 \tag{2.1.13}$$

With a dimensionless velocity $u^\alpha = U_\alpha / n^0$, we get from (2.1.13) and (2.1.10), respectively:

$$u^\alpha u_\alpha = -1; \qquad (\rho^0 u^\alpha)|_\alpha = 0 \tag{2.1.14}$$

another form of the law of the conservation of mass.

To obtain the laws of conservation of momentum and energy we use β_4 instead of β_3 in (1.4.18) to derive the formula for the force per unit mass in the $\{Y\}$-system, $\mathscr{F}^{*j}(Y)$:

$$\mathscr{F}^{*j}(Y) = m_0^{-1} c\beta_4^{-1} F^j(Y); \qquad v_j F^j(Y) = c^{-2} I_0^2 F^4(Y) \tag{2.1.15}$$

expressing v_j in terms of ξ_j, taking mean values and using (2.1.11) and $(2.1.15)_1$ gives:

$$n\bar{F}^j(Y)\bar{u}_j = c^{-2} I_0^2 n\langle \bar{F}^4(Y)\rangle \tag{2.1.16}$$

$$\mathscr{F}^{*j}\bar{u}_j = c^{-2} I_0^2 \mathscr{F}^{*4}; \qquad \mathscr{F}^{*4} = m_0^{-1} c\beta_4^{-1}\langle \bar{F}^4(Y)\rangle \tag{2.1.17}$$

Starting from the functional $\int \bar{F}^j(Y) V_j \, d\mu$, using (1.5.10), the definition of the mean value, (2.1.5) and (2.1.9) and some operations, we obtain:

$$\bar{F}^j(Y) U_j + \langle \bar{F}^4\rangle U_4 = 0 \qquad \text{or} \qquad \mathscr{F}^{*\alpha} u_\alpha = 0 \tag{2.1.18}$$

The functional below (2.1.17) with the use of (2.1.13) and $(2.1.17)_2$, is equal to $c^{-2} I_0^2 n^0 m_0 \mathscr{F}^{*4}$. Using (2.1.7) and (2.1.8), transformation of coordinates (1.5.7), (2.1.15) and $(2.1.17)_2$, and the last expression for the functional, after some transformations, we get the equation for the energy-momentum tensor:

$$T^{\alpha\beta}|_\beta = \rho^0 \mathscr{F}^{*\alpha}; \qquad T^{\alpha\beta} = m_0 c^2 \int V^\alpha V^\beta \, d\mu \tag{2.1.19}$$

We may combine both sides of $(2.1.19)_1$ introducing a new tensor for the external forces $\mathscr{F}^{*\alpha}$:

$$\rho^0 \mathscr{F}^{*\alpha} = \Pi^{\alpha\beta}|_\beta; \qquad T^{*\alpha\beta} = T^{\alpha\beta} - \Pi^{\alpha\beta}; \qquad T^{*\alpha\beta}|_\beta = 0 \tag{2.1.20}$$

2.2. *Characteristic Properties*

Certain aspects of the present approach are identical to those derived by Taub (1948); thus the specific internal energy per unit mass, ε, is defined in the $\{Y\}$-system with the use of (2.1.13) from the equation:

$$m_0^2(\rho^0)^{-2} T_{\alpha\beta} U^\alpha U^\beta = T_{\alpha\beta} u^\alpha u^\beta = \rho^0(c^2 + \varepsilon) \tag{2.2.1}$$

where $T_{\alpha\beta}$ is the covariant tensor associated with the contravariant tensor (2.1.19). The internal energy per unit mass of the fluid is a function of the pressure and the rest density. Taub (1948) derived a lower bound for ε in the $\{Y\}$-space-time coordinates:

$$\varepsilon \geqslant \tfrac{3}{2} p(\rho^0)^{-1} + c^2\{[1 + \tfrac{9}{4}(pc^{-2}(\rho^0))^2]^{1/2} - 1\} \tag{2.2.2}$$

The inequality (2.2.2) imposes a restriction on the possible kinds of function $\varepsilon(p, \rho^0)$ in the relativistic kinetic theory of gases in contradiction to the macroscopic theory, where E may be any function of p and ρ^0. It is valid in the present case.

2.3. *Rankine-Hugoniot Equations*

We consider the motion of a perfect gas subject to no external forces. Following Taub (1948) we assume in the $\{Y\}$-space-time the forms [see (2.1.20)]:

$$T^{\alpha\beta} = \rho^0 c^2[1 + c^{-2}(\varepsilon + p(\rho^0)^{-1})] u^\alpha u^\beta + pG^{\alpha\beta}, \qquad \Pi^{\alpha\beta} = U_0 G^{\alpha\beta} \quad (2.3.1)$$

where p denotes the static pressure, and U_0 [see (1.1.5)] the value of the gravitational potential at a certain point '0'. From this we get:

$$T^{*\alpha\beta} = \rho^0 c^2[1 + c^{-2}(\varepsilon + p(\rho^0)^{-1})] u^\alpha u^\beta + (p - U_0) G^{\alpha\beta} \quad (2.3.2)$$

with $(2.1.20)_3$ and $(2.1.14)_2$ retaining their validity. Due to the fact that U is assumed to be a piece-wise (point-wise) constant function, i.e., $U_0 =$ constant, the present flow reduces to the flow discussed by Taub (1948), with the specific entropy, S, as measured by an observer at rest with respect to the gas, being constant along a stream-line.

Let us consider a one-dimensional motion in $\{Y^1, Y^4 = t\}$-space-time and remodel $(2.1.11)_1$ using $(2.1.9)_2$ and (2.1.12), and the definition of u^α below (2.1.13):

$$\bar{u}^j = I_0(1 - \bar{u}^2 I_0^{-2})^{1/2} u^j; \qquad u^j = \bar{u}^j I_0^{-1}(1 - \bar{u}^2 I_0^{-2})^{-1/2};$$
$$\bar{u}^2 = G_{jk}\bar{u}^j \bar{u}^k \quad (2.3.3)$$

With $\bar{u}^1 I_0^{-1} \equiv u$, we get from (2.1.14) and $(2.3.3)_2$:

$$u^1 = u(1 - u^2)^{-1/2}; \qquad u^2 = u^3 = 0; \qquad u^4 = cI_0^{-1}(1 - u^2)^{-1/2} \quad (2.3.4)$$

Inserting (2.3.4) into $(2.1.14)_2$ and $(2.1.20)_3$ we obtain in the $\{y\}$-space-time [the transformation $Y \leftrightarrow y$ is given in (1.5.7)], after carrying out the indicated differentiation:

$$(1 - u^2)[I_0^{-1}(\rho^0)^{-1}\rho^0_{,t} + u(\rho^0)^{-1}\rho^0_{,y}] + I_0^{-1} uu_{,t} + u_{,y} = 0 \quad (2.3.5)$$

$$u(1 - u^2)[I_0^{-1}\mu^{-1}\mu_{,t} + \mu^{-1}uu_{,y}] + I^{-1}u_{,t} + uu_{,y} + (1 - u^2)^2 .$$
$$.(\rho^0)^{-1}c^{-2}\mu^{-1}p_{,y} = 0 \quad (2.3.6)$$

$$\mu = 1 + c^{-2}[\varepsilon + p(\rho^0)^{-1}] \quad (2.3.7)$$

Introducing auxiliary variables

$$\alpha^2 = \rho^0 \mu^{-1} d\mu/d\rho^0; \qquad \varphi = \int \alpha(\rho^0)^{-1} d\rho^0 \quad (2.3.8)$$

equations (2.3.5) and (2.3.6) transform into:

$$(1 - u^2)[I_0^{-1}\varphi_{,t} + u\varphi_{,y}] + \alpha[I_0^{-1}uu_{,t} + u_{,y}] = 0 \quad (2.3.9)$$

$$\alpha(1 - u^2)[uI_0^{-1}\varphi_{,t} + \varphi_{,y}] + [I_0^{-1}u_{,t} + uu_{,y}] = 0 \quad (2.3.10)$$

Addition and subtraction of (2.3.9) and (2.3.10) yield:

$$(1 - u^2) D_+\varphi + D_+ u = 0; \qquad (1 - u^2) D_-\varphi - D_- u = 0 \quad (2.3.11)$$

$$D_\pm = (1 \pm \alpha u) I_0^{-1}\,\partial/\partial t \pm (\alpha \pm u)\,\partial/\partial y \quad (2.3.12)$$

Equations (2.3.11) and (2.3.12) are identical to those derived by Taub (1948) referring to the velocity of light in a vacuum. Beginning from this point both formalisms referring to the velocity of light in a vacuum and in a non-vacuum, respectively, are identical, or analogous, i.e., the formalism given by Taub (1948) is applicable directly to the present case. In particular:

(i) The relativistic analogs of the Riemann functions, r, s, which occur in the classical theory of the propagation of one-dimensional waves of finite amplitude:

$$D_+ r = 0; \qquad D_- s = 0; \qquad r, s = \varphi \pm \ln [(1+u)(1-u)^{-1}]^{1/2} \quad (2.3.13)$$

(ii) The functions r, s, are constant along the curves, respectively:

$$(dy/dt)_{\mathrm{I,\,II}} = \pm I_0(\alpha \pm u)(1 \pm \alpha u)^{-1} \quad (2.3.14)$$

(iii) A disturbance propagates as a progressive wave if either r or s is constant. Using this idea and the formalism of Taub (1948) one can derive the magnitude of the velocity of the propagation of weak disturbances for $u \to 0$. It is equal to α, which may be expressed as aI_0^{-1}, i.e., α is the velocity of sound in units where the velocity of the signal, I_0, is unity.

(iv) For a high temperature ($p(c^2\rho^0)^{-1}$ large), $\alpha \to (\gamma - 1)^{1/2}$, which implies that for $\gamma > 2$ sound waves should propagate with a velocity greater than I_0, which would be impossible. The relativistic Rankine-Hugoniot equations derived by Taub (1948) remain valid in the present case. The variables ρ^0, u^α, p, ε, μ are subject to jump discontinuities with I_0, U_0 remaining constant. Let the Y^1-axis be perpendicular to the normal shock.

Mass: $$\rho_+^0 u_+(1-u_+^2)^{-1/2} = \rho_-^0 u_-(1-u_-^2)^{-1/2} = M \quad (2.3.15)$$

(from) Momentum:

$$M = c^{-1}\{(p_+ - p_-)[\mu_- . (\rho_-^0)^{-1} - \mu_+ . (\rho_+^0)^{-1}]^{-1}\}^{1/2} \quad (2.3.16)$$

Energy: $$M^2 c^2(\mu_+^2 - \mu_-^2) = M^2(p_+ - p_-)[\mu_+(\rho_+^0)^{-1} + \mu_-(\rho_-^0)^{-1}] \quad (2.3.17)$$

Assume that the gaseous medium moves from right to left across a normal, fixed shock. Variables on the right-hand side of the shock are denoted by subscript (–), whereas those on the left-hand side are denoted by subscript (+). Introduce the quantities:

$$\xi = p_+(p_-)^{-1}; \qquad \eta = \rho_+^0(\rho_-^0)^{-1}; \qquad \beta = \gamma_+(\gamma_+ - 1)^{-1} c^{-2} p_-(\rho_-^0)^{-1} \quad (2.3.18)$$

which when inserted into (2.3.7) with the use of the relativistic definition of the internal energy per unit mass $\varepsilon = (\gamma - 1)^{-1} p(\rho^0)^{-1}$, give:

$$\mu_+ = 1 + \beta\xi\eta^{-1}; \qquad \mu_- = 1 + (\gamma_-)(\gamma_+)^{-1}(\gamma_+ - 1)(\gamma_- - 1)^{-1}\beta \quad (2.3.19)$$

Following Taub (1948) assume γ_+ to be a constant. (2.3.17) then takes the

TABLE 1

Variable	$\{Y^{\cdot}\}$-space-time (Taub, 1948)	The present work $\{Y\}$-space-time
Velocity vector	$V^{\cdot j} = \xi^{\cdot j} c^{-1};\ V^{\cdot 4} = (1 + \xi^{\cdot 2} c^{-2})^{1/2}$	$V^{j} = \xi^{j} c^{-1};\ V^{4} = c I_0^{-1}(1 + \xi^{2} c^{-2})^{1/2}$
Mass-current vector	$U^{\cdot \alpha} = \int V^{\cdot \alpha} d\mu^{\cdot};\ d\mu^{\cdot} = f(1 + \xi^{\cdot 2} c^{-2})^{-1/2} d_3 \xi^{\cdot}$	$U^{\alpha} = \int V^{\alpha} d\mu;\ d\mu = f(1 + \xi^{2} c^{-2})^{-1/2} d_3 \xi$
Average velocity	$\bar{u}^{\cdot j} = n^{\cdot -1} \int v^{\cdot j} f\, d_3 \xi^{\cdot} = n^{\cdot -1} \int \xi^{\cdot j} d\mu^{\cdot}$	$\bar{u}^{j} = n^{-1} \int v^{j} f\, d_3 \xi = n^{-1} \int c^{-1} I_0 \xi^{j} d\mu$
	$\bar{u}^{\cdot 2} = G_{jk} \bar{u}^{\cdot j} \bar{u}^{\cdot k}$	$\bar{u}^{2} = G_{jk} \bar{u}^{j} \bar{u}^{k}$
Dimensionless one-dimensional average velocity	$u^{\cdot} = \bar{u}^{\cdot 1} c^{-1}$	$u = \bar{u}^{1} I_0^{-1}$
Number density	$n^{\cdot} = \int f(Y^{\cdot j}, \xi^{\cdot j}, t)\, d_3 \xi^{\cdot}$	$n = \int f(Y^{j}, \xi^{j}, t)\, d_3 \xi$
Number density at rest	$n^{\cdot 0} = n^{\cdot}(1 - \bar{u}^{\cdot 2} c^{-2})^{1/2} = (-U^{\cdot \alpha} U_{\alpha}^{\cdot})^{1/2}$	$n^{0} = n(1 - \bar{u}^{2} I_0^{-2})^{1/2} = (-U^{\alpha} U_{\alpha})^{1/2}$
Dimensionless velocity vector	$u^{\cdot \alpha} = (n^{\cdot 0})^{-1} U^{\cdot \alpha}$	$u^{\alpha} = (n^{0})^{-1} U^{\alpha}$
Density at rest	$\rho^{\cdot 0} = n^{\cdot 0} m_0$	$\rho^{0} = n^{0} m_0$
Pressure	$p^{\cdot}(p_1^{\cdot})^{-1} = [\rho^{\cdot 0}(\rho^{\cdot 0})^{-1}]^{\gamma}$	$p(p_1)^{-1} = [\rho^{0}(\rho_1{}^{0})^{-1}]^{\gamma}$
Internal energy	$\varepsilon^{\cdot} = (\gamma - 1)^{-1} [p^{\cdot}(\rho^{\cdot 0})^{-1}]$	$\varepsilon = (\gamma - 1)^{-1} [p(\rho^{0})^{-1}]$
μ	$\mu^{\cdot} = 1 + c^{-2}[\varepsilon^{\cdot} + p^{\cdot}(\rho^{\cdot 0})^{-1}]$	$\mu = 1 + c^{-2}[\varepsilon + p(\rho^{0})^{-1}]$
Dimensionless velocity of sound	$\alpha^{\cdot 2} = c^{-2} \gamma p^{\cdot}(\rho^{\cdot 0})^{-1} [1 + c^{-2} \gamma(\gamma - 1)^{-1} p^{\cdot}(\rho^{\cdot 0})^{-1}]^{-1}$	$\alpha^{2} = c^{-2} \gamma p(\rho^{0})^{-1} [1 + c^{-2} \gamma(\gamma - 1)^{-1} p(\rho^{0})^{-1}]^{-1}$
Velocity of sound	$a^{\cdot} = \alpha^{\cdot} c$	$a = \alpha I_0$
Shock velocity relative to the gas	$\dfrac{u_{-}^{\cdot}}{(1 - u^{\cdot 2})^{1/2}} = \dfrac{[(\gamma_{+} - 1) \beta^{\cdot}(\xi^{\cdot} - 1)]^{1/2}}{\{\gamma_{+}[\mu_{-}^{\cdot} - \mu_{+}^{\cdot}(\eta^{\cdot})^{-1}]\}^{1/2}}$	$\dfrac{u_{-}}{(1 - u_{-}^{2})^{1/2}} = \dfrac{[(\gamma_{+} - 1) \beta(\xi - 1)]^{1/2}}{\{\gamma_{+}[\mu_{-} - \mu_{+} \eta^{-1}]\}^{1/2}}$

following form, with $\gamma_+ = \gamma_-$, $\mu_- = 1 + \beta$, from (2.2.2), $\varepsilon > 0$, $\frac{5}{3} \geqslant \gamma_+ > 1$:

$$\beta\eta^{-1} = \{R - [(\gamma_+ + 1)\,\xi + (\gamma_+ - 1)]\}\,\{2\xi[\xi + (\gamma_+ - 1)]\}^{-1} \quad (2.3.20)$$

$$R = \{(\gamma_+ - 1)^2\,(\xi - 1)_+{}^2\,4\xi(\xi + \gamma_+ - 1)\,[\gamma_+\mu_-{}^2 + \beta\mu_-(\gamma_+ - 1)\,(\xi - 1)]\}^{1/2} \quad (2.3.21)$$

A combination of (2.3.15) and (2.3.16) gives:

$$u_-(1 - u_-{}^2)^{-1/2} = [(\gamma_+ - 1)\,\beta(\xi - 1)]^{1/2}\,[\gamma_+(\mu_- - \mu_+\eta^{-1})]^{-1/2} \quad (2.3.22)$$

Suppose we wish to have a system in which the medium on the right-hand side of the shock is at rest and the shock moves into it. To achieve this we may superimpose upon the entire medium-shock-system the velocity of the magnitude $\bar{u}_-{}^1$ from left to right. The medium on the right-hand side will be at rest and the velocity of the shock will be $\bar{u}_-{}^1$.

A transformation from the $\{Y\}$-space-time in the present work, with the I_0 reference velocity to the $\{Y'\}$-space-time given by Taub (1948) with the c-reference velocity is of the form:

$$Y'^j = Y^j; \qquad Y'^4 = c^{-1}\,I_0\,Y^4 \quad (2.3.23)$$

The relations between variables in the both reference frames are presented in Tables 1 and 2. The variables $\rho^0, p, \varepsilon, \mu$ are invariant under (2.3.23) since

TABLE 2. Relation between variables in $\{Y'\}$- and $\{Y\}$-space-times

$\xi'^j = \xi^j;\ V'^j = V^j;\ V'^4 = c^{-1}\,I_0\,V^4$
$U'^j = U^j;\ U'^4 = c^{-1}\,I_0\,U^4$
$\bar{u}'^j = cI_0^{-1}\,\bar{u}^j$
$\bar{u}'^2 = c^2\,I_0^{-2}\,\bar{u}^2$
$u' = u$
$n' = n$
$n'^0 = n^0$
$u'^j = u^j;\ u'^4 = c^{-1}\,I_0\,u^4$
$\rho'^0 = \rho^0$
$p' = p$
$\varepsilon' = \varepsilon$
$\mu' = \mu$
$\alpha' = \alpha$
$a' = cI_0^{-1}\,a$
$u_-' = u_-;\ \bar{u}_-'^1 = cI_0\,\bar{u}_-{}^1$
$\bar{u}_-'^1 > \bar{u}_-{}^1$

the factors $(1 - u'^2)^{1/2}$ and $(1 - u^2)^{1/2}$ are equal and the distribution function $f(Y^j, \xi^j, t)$ is also invariant under (2.3.23). Formally, only $\bar{u}_{\pm}{}^1$ and a are effected by introducing I_0 in place of c. The transformation due to the superimposing the velocity $\bar{u}_-{}^1$, $\{Y^*\} \to \{Y\}$, is of the Lorentz type:

$$Y^{*1} = \{Y^1 - c^{-1}\bar{u}_-{}^1 t\}(1 - (\bar{u}_-{}^1)^2 I_0^{-2})^{-1/2}; \qquad Y^{*2} = Y^2;$$
$$Y^{*3} = Y^3; \qquad t^* = (t - \bar{u}_-{}^1 c I_0^{-2} Y^1)(1 - (\bar{u}_-{}^1)^2 I_0^{-2})^{-1/2}$$

which leave $(d\tau)^2$ invariant. The velocity $\bar{u}_-{}^1$ is considered to be momentarily (piece-wise) constant for the above transformation to be valid. In the final forms of equations the star in Y^{*j} is omitted.

3. *Relativistic Hydrodynamic Equations in Riemannian Space-Time*

3.1. *Introductory Remarks*

In this section we derive the relativistic hydrodynamical equations in the Riemannian space-time $\{x\}$ with the reference velocity $I = I(x^j)$. We operate in a system of curvilinear coordinates $\{x^j\}$, embedded in the $\{x\}$ space-time. We shall investigate the relation between the forms of hydrodynamic equations in both Euclidean and Riemannian space.

3.2. *Boltzmann Equation*

We begin with the Boltzmann equation in the Riemannian space-time:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^j}\frac{dx^j}{dt} + \frac{\partial f}{\partial z^j}\frac{dz^j}{dt} = \Delta_e f \tag{3.2.1}$$

where $f = f(x^j, z^j, t)$, and the symbol d/dt denotes the absolute derivative.† From (1.4.3) we calculate q^j in terms of z^j:

$$q^j = c^{-1} I z^j (1 + c^{-2} z^2)^{-1/2}; \qquad c^{-2} z^2 = a_{jk} z^j z^k \tag{3.2.2}$$

From (1.4.12) with $\delta z^j/\delta t = c^{-2} m_0^{-1} \bar{F}^j(x)$ one obtains:

$$dz^j/dt = c^{-2} m_0^{-1}\{\bar{F}^j(x) - \tfrac{1}{2} M a^{kj}(I^2)_{,k}\} \tag{3.2.3}$$

which inserted into (3.2.1) with $dx^j/dt = q^j$, gives:

$$Df \equiv \partial f/\partial t + c^{-1} I z^j (1 + z^2 c^{-2})^{-1/2} \partial f/\partial x^j$$
$$+ c^{-2} m_0^{-1}[\bar{F}^j(x) - \tfrac{1}{2} M a^{jk}(I^2)_{,k}] \partial f/\partial z^j = \Delta_e f \tag{3.2.4}$$

Multiply (3.2.4) by any transport quantity $\Phi(x^j, z^j, t)$, integrate over the entire (z^1, z^2, z^3)-space, apply integration by parts, with products $(f\Phi)$ tending to zero for z^j approaching to $\pm\infty$, express M in terms of z^2 and I^2:

$$M = m_0 c I^{-1}(1 - q^2 I^{-2})^{-1/2} = m_0 c I^{-1}(1 + c^{-2} z^2)^{1/2} \tag{3.2.5}$$

One obtains:

† The total (ordinary) derivative of a scalar is identical to its absolute derivative.

$$\begin{aligned}\int \Phi(Df)\,d_3 z = {} & (n\langle\Phi\rangle)_{,t} + (c^{-1}I)\,[n\langle\Phi z^j.(1+c^{-2}z^2)^{-1/2}\rangle]_{,j} \\ & + (c^{-1}I)_{,j}\,[n\langle\Phi z^j(1+c^{-2}z^2)^{-1/2}\rangle] \\ & - \{n\langle\Phi_{,t}\rangle + n\langle[(c^{-1}I)\,\Phi z^j(1+c^{-2}z^2)^{-1/2}]_{,j}\rangle \\ & + n\langle c^{-2}m_0^{-1}\bar{F}^j(x)\,\Phi_{,zj}\rangle \\ & - n\langle(1+c^{-2}z^2)^{1/2}\,\Phi_{,zj}\rangle\,a^{jk}(c^{-1}I)_{,k} \\ & + n\langle c^{-2}m_0^{-1}\,\Phi\bar{F}^{\,j}_{,zj}\rangle \\ & - n\langle\Phi z^j(1+c^{-2}z^2)^{-1/2}\rangle\,(c^{-1}I)_{,j}\} \\ = {} & \int \Phi\Delta_e f.\,d_3 z \end{aligned} \tag{3.2.6}$$

To obtain the summational invariants we insert

$$\Phi = \Psi^0(x) = m_0, \qquad \Phi = \Psi^j(x) = m_0\,z^j,$$

$$\Phi = \Psi^4(x) = c^2\,m_0(cI^{-1})\,(1+c^{-2}z^2)^{1/2} = Mc^2$$

The function $\Psi^0 = m_0 = \text{constant}$, gives the law of the conservation of mass:

$$m_0\{n_{,4} + c^{-1}I[n\langle z^j(1+c^{-2}z^2)^{-1/2}\rangle]_{,j} + (c^{-1}I)_{,j}\,n\langle z^j(1+c^{-2}z^2)^{-1/2}\rangle\} = c^{-2}\,n\langle\bar{F}^j_{,j}\rangle \tag{3.2.7}$$

With (1.5.1), and the assumption that the forces are independent of the velocity z^j and with $I = I(X^j)$, equation (3.2.7) takes the form:

$$m_0\left\{c^{-1}I\left[\int V^4\,d\mu\right]_{,4} + c^{-1}I\left[\int V^j\,d\mu\right]_{,j} + (c^{-1}I)_{,j}\int V^j\,d\mu\right\} = 0 \tag{3.2.8}$$

$$d\mu = (1+c^{-2}z^2)^{-1/2}\,f\,d_3 z \tag{3.2.9}$$

Using the mass current vector $U^\alpha(X) = \int V^\alpha(X)\,d\mu$, we get:

$$m_0\{U^j_{,j} + U^4_{,4} + \tfrac{1}{2}[\ln(c^{-2}I^2)]_{,j}\,U^j\} = 0 \tag{3.2.10}$$

or

$$m_0\,U^\alpha|_\alpha = 0 \tag{3.2.11}$$

Introducing the average velocity:

$$\bar{w}^j = n^{-1}\int q^j f\,d_3 z; \qquad \bar{w}^2 = A_{jk}\,\bar{w}^j\,\bar{w}^k \tag{3.2.12}$$

we calculate:

$$1 - \bar{w}^2 I^{-2} = -n^2\,U^\alpha\,U_\alpha \tag{3.2.13}$$

We define the number density measured by an observer moving with $\bar{w}^j$ with respect to the fixed coordinate system $\{X^j\}$:

$$(n^0(X))^2 = n^2(X)\,(1 - \bar{w}^2 I^{-2}) = -U^\alpha(X)\,U_\alpha(X) \tag{3.2.14}$$

with the corresponding density $\rho^0 = n^0 m_0$. Introducing the dimensionless velocity $w^\alpha = (n^0)^{-1}\,U^\alpha$, we obtain from (3.2.14), $w_\alpha w^\alpha = -1$, and from

(3.2.11) $(\rho^0 w^\alpha)|_\alpha = 0$, which is another representation of the law of the conservation of mass.

Introducing the force vector we get:

$$\mathscr{F}^{*\alpha} = m_0^{-1} c I^{-1}(1 - \bar{w}^2 I^{-2})^{-1/2} \langle \bar{F}^\alpha(X) \rangle; \qquad \langle \bar{F}^j \rangle = \bar{F}^j \quad (3.2.15)$$

$$\mathscr{F}^{*\alpha}(X) w_\alpha = 0; \qquad \int \bar{F}^j(X) V_j \, d\mu = c^{-2} I^2 n^0 m_0 \mathscr{F}^{*4}(X) \quad (3.2.16)$$

Inserting $\Psi^k = m_0 z^k$ into (3.2.6), assuming again that $\bar{F}^j$ is independent of z^j and using the transformation (1.5.1), with $I = I(X^j)$, gives:

$$I\Big[m_0 \int V^k V^4 d\mu\Big]_{,4} + I\Big[m_0 \int V^j V^k d\mu\Big]_{,Xj}\Big\}$$

$$+ (c^{-2} I^2)(c^{-2} I)_{,Xj} m_0 A^{ik} \int V^4 V^4 d\mu + I_{,Xj} m_0 \int V^j V^k d\mu$$

$$= c^{-1} \rho m_0{}^{-1} \bar{F}^k(X); \quad \rho = n m_0 \quad (3.2.17)$$

With $T^{k\alpha} = m_0 c^2 \int V^k V^\alpha d\mu$, (3.2.17) becomes:

$$T^{4k}_{,4} + T^{jk}_{,j} + \tfrac{1}{2}(c^{-2} I^2)_{,i} A^{ik} T^{44} + \tfrac{1}{2}[\ln(c^{-2} I^2)]_{,j} T^{jk}$$

$$= \rho^0 \mathscr{F}^{*k}(X) = c I_0^{-1} \rho m^{-1} \bar{F}^k(X) \quad (3.2.18)$$

In an exactly similar way, inserting Ψ^4 into (3.2.6), using $\{x\} \to \{X\}$, $T^{4\sigma} = m_0 c^2 \int V^4 V^4 d\mu$, we obtain:

$$T^{4j}_{,j} + T^{44}_{,4} + \tfrac{3}{2}[\ln(c^{-2} I^2)]_{,j} T^{4j} = \rho^0 \mathscr{F}^{*4}(X) \quad (3.2.19)$$

which combined with (3.2.18) gives:

$$T^{\alpha\beta}|_\beta = \rho^0 \mathscr{F}^{*\alpha}; \qquad T^{\alpha\beta} = m_0 c^2 \int V^\alpha V^\beta d\mu \quad (3.2.20)$$

From $\rho^0 \mathscr{F}^{*\alpha} = \Pi^{\alpha\beta}|_\beta$, $T^{*\alpha\beta} = T^{\alpha\beta} - \Pi^{\alpha\beta}$, we get:

$$T^{*\alpha\beta}|_\beta = 0 \quad (3.2.21)$$

The internal energy of the medium per unit mass, ε, is expressed by a formula similar to that one derived in the Euclidean space:

$$T_{\alpha\beta} w^\alpha w^\beta = \rho^0(c^2 + \varepsilon) \quad (3.2.22)$$

Following Taub (1948) we may assume:

$$T^{\alpha\beta} = \rho^0 c^2[1 + c^{-2}(\varepsilon + p(\rho^0)^{-1})] w^\alpha w^\beta + p A^{\alpha\beta} \quad (3.2.23)$$

or with $\Pi^{\alpha\beta} = \Psi A^{\alpha\beta}$:

$$T^{*\alpha\beta} = \rho^0 c^2[1 + c^{-2}(\varepsilon + p(\rho^0)^{-1})] w^\alpha w^\beta + (p - \Psi) A^{\alpha\beta} \quad (3.2.24)$$

Insert (3.2.24) into (3.2.21), multiply the result by $(-w_\alpha)$, use the result that $w_\alpha w^\alpha = -1$, $(\rho^0 w^\alpha)|_\alpha = 0$, $(3.2.16)_1$, and obtain:

$$\{\varepsilon_{,\beta} w^\beta + p[(\rho^0)^{-1}]_{,\beta} w^\beta\} = d\varepsilon + p\, d[(\rho^0)^{-1}] = 0 \quad (3.2.25)$$

with $d\varepsilon = \varepsilon_{,\beta} w^{\beta}$, $d[(\rho^0)^{-1}] = [(\rho^0)^{-1}]_{,\beta} w^{\beta}$. This expresses the first law of thermodynamics with the elementary energy (heat) input into the system, dQ, being equal to zero. With S denoting the specific entropy and θ the absolute temperature, we get from (3.2.25)

$$dQ = \theta\, dS = d\varepsilon + p\, d[(\rho^0)^{-1}] = 0$$

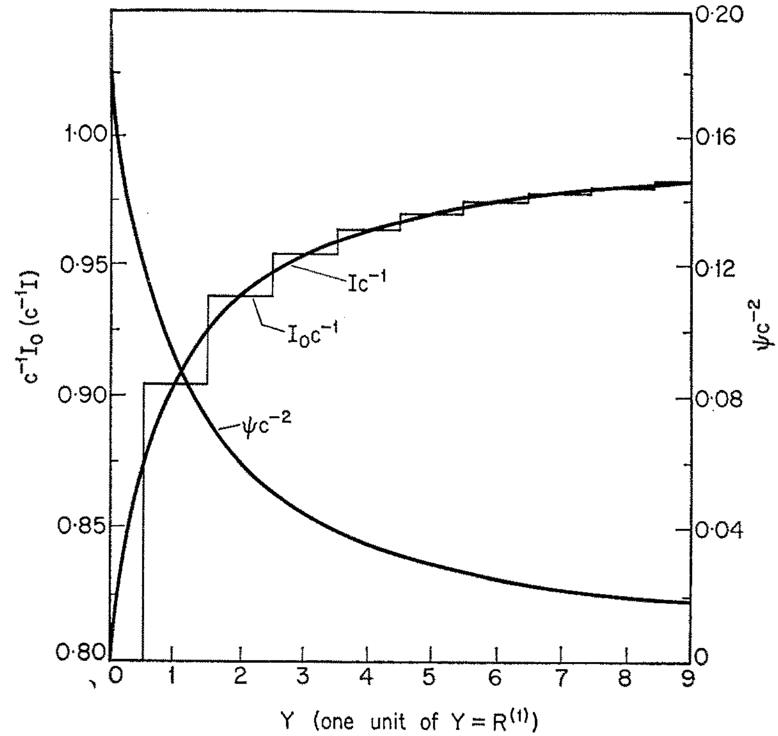

Figure 1.—The dimensionless gravitational potential $c^{-2}\psi$ and the dimensionless velocity $c^{-1}I_0(c^{-1}I)$ as functions of the Y-coordinate.

3.3. *One-Dimensional Flow*

In $(X^1, X^4 = t)$ system of coordinates (3.2.12) with U^{α} [above (3.2.10)], (3.2.14) and the expression for w^{α} takes the form:

$$w^1 = \bar{w}^1 I^{-1}[1 - (\bar{w})^2 I^{-2}]^{-1/2}; \qquad (\bar{w})^2 = A_{jk}\,\bar{w}^j \bar{w}^k \tag{3.3.1}$$

or with $\bar{w}^1 I^{-1} = w$, we get:

$$w^1 = w(1 - w^2)^{-1/2} \tag{3.3.2}$$

which inserted into $w_{\alpha} w^{\alpha} = -1$, gives:

$$w^4 = cI^{-1}(1 - w^2)^{-1/2} \tag{3.3.3}$$

Obviously, the above equations can be easily expressed in the (x, t)-space-time, by the simple transformation $(X) \leftrightarrow (x)$. Inserting (3.3.2), (3.3.3) into $(\rho^0 w^\alpha)|_\alpha = 0$, and expanding, gives in the (x, t)-space-time:

$$(1 - w^2)[I^{-1}(\rho^0)^{-1}\rho^0_{,t} + w(\rho^0)^{-1}\rho^0_{,x}] + I^{-1}ww_{,t} + w_{,x} = -\tfrac{1}{2}w(1 - w^2)(\ln I^2)_{,x} \quad (3.3.4)$$

Analogously, inserting the same expression into (3.2.21), expressing $\mathscr{F}^{*\alpha}$ in terms of the gradient of Ψ, gives:

$$w(1 - w^2)[I^{-1}\mu^{-1}\mu_{,t} + w\mu^{-1}\mu_{,x}] + I^{-1}w_{,t} + ww_{,x} + (1 - w^2)(\rho^0)^{-1}c^{-2}\mu^{-1}p_{,x} = -\tfrac{1}{2}(1 - w^2)(\ln I^2)_{,x} + (1 - w^2)(\rho^0)^{-1}\mu^{-1}\Psi_{,x} \quad (3.3.5)$$

$$\mu = 1 + c^{-2}[\varepsilon + p(\rho^0)^{-1}] \quad (3.3.6)$$

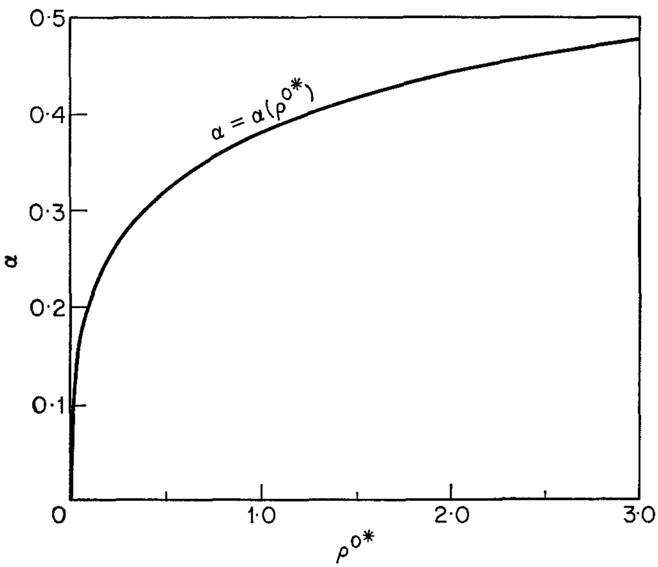

Figure 2.—The dimensionless sound velocity α as a function of the normalized density ρ^{0*}.

Let us introduce auxiliary variables (2.3.8) into (3.3.4) and (3.3.5), giving:

$$(1 - w^2)(I^{-1}\varphi_{,t} + w\varphi_{,x}) + \alpha(I^{-1}ww_{,t} + w_{,x}) = -\tfrac{1}{2}\alpha w(1 - w^2)(\ln I^2)_{,x} \quad (3.3.7)$$

$$\alpha(1 - w^2)(I^{-1}w\varphi_{,t} + \varphi_{,x}) + (I^{-1}w_{,t} + ww_{,x}) = (1 - w^2)[(\rho^0)^{-1}c^{-2}\mu^{-1}(1 - w^2)\Psi_{,x} - \tfrac{1}{2}(\ln I^2)_{,x}] \quad (3.3.8)$$

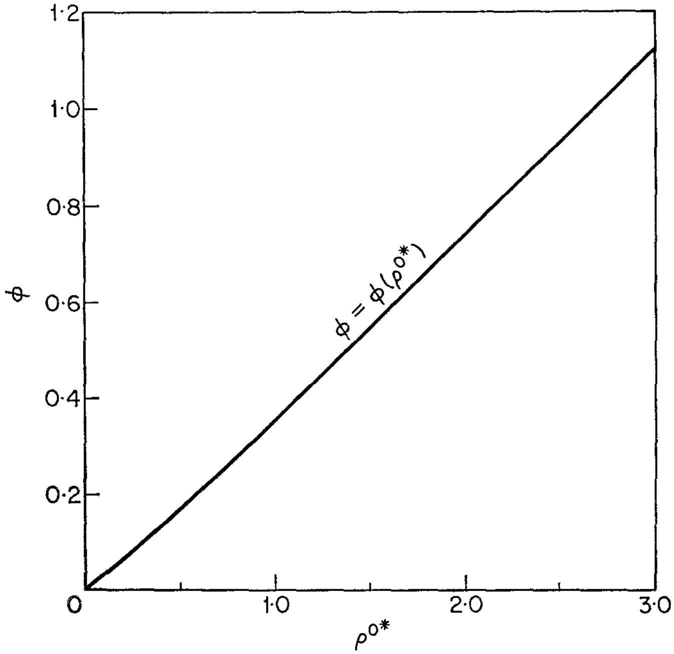

Figure 3.—The quantity ϕ as a function of the normalized density ρ^{0*}.

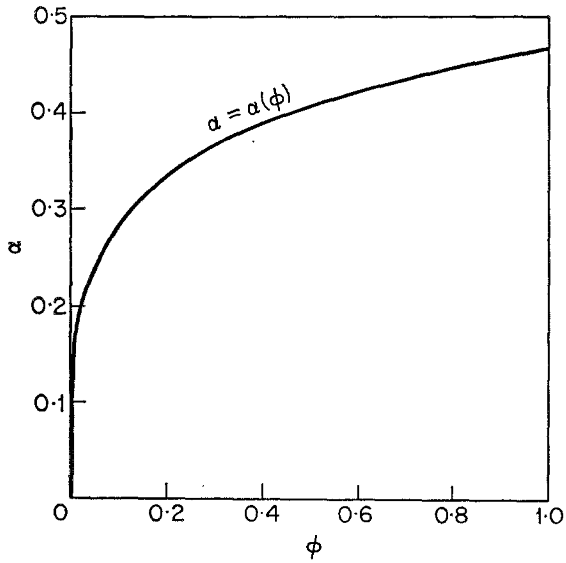

Figure 4.—The dimensionless sound velocity α as a function of the quantity ϕ.

Adding and subtracting (3.3.7) and (3.3.8) gives:

$$(1-w^2)\,D_+^{(1)}\varphi + D_+^{(1)}w = L^{(1)};\quad (1-w^2)\,D_-^{(1)}\varphi - D_-^{(1)}w = L^{(2)} \tag{3.3.9}$$

$$D_\pm = (1 \pm \alpha w)\,I^{-1}\,\partial/\partial t \pm (\alpha \pm w)\,\partial/\partial x \tag{3.3.10}$$

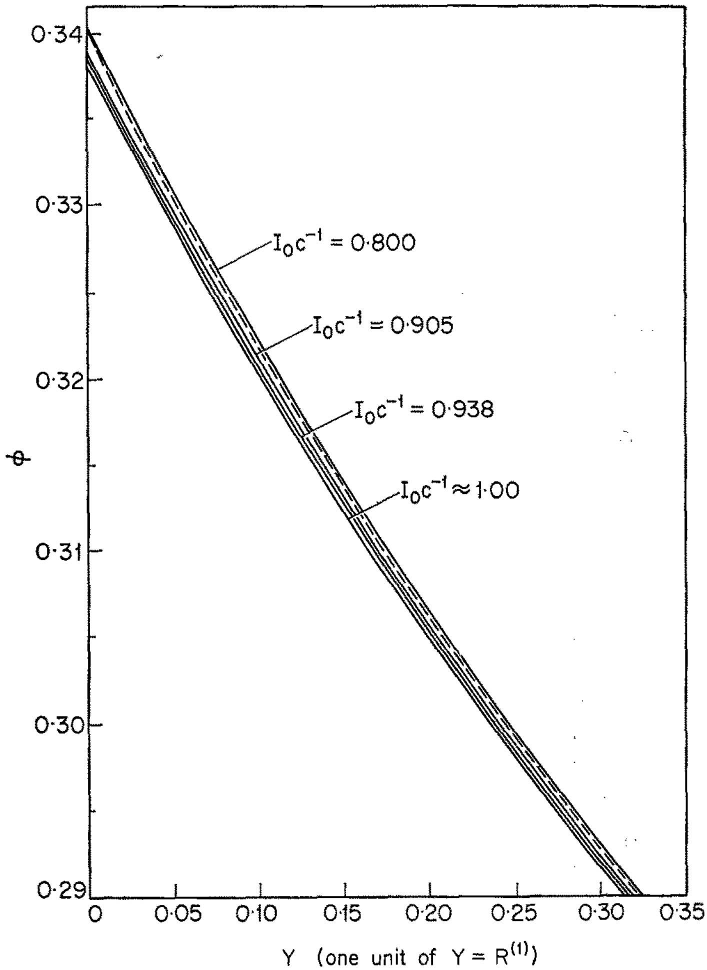

Figure 5.—Solution of equations (2.3.9) and (2.3.10) for ϕ at the particular instant $t = t_0 = 0{\cdot}4$ for different values of the parameter $I_0\,c^{-1}$ and the corresponding approximate solution for ϕ (shown in dashed line) in the range of Y [0,0·325].

$$L^{(1)}, L^{(2)} = \pm(1-w^2)\,[\tfrac{1}{2}(1 \pm \alpha w)(\ln I^2)_{,x} - (\rho^0)^{-1}\,c^{-2}\,\mu^{-1}(1-w^2)\,\Psi_{,x}] \tag{3.3.11}$$

The one-dimensional equations in the Riemannian space-time derived in the present chapter are reducible to formulae derived in Section 2.3 by setting: $I = I_0$ = constant, $\Psi = \Psi_0$ = constant and all the derivatives of I_0, Ψ_0 equal to zero. Mathematically, operating in the flat space, we find only

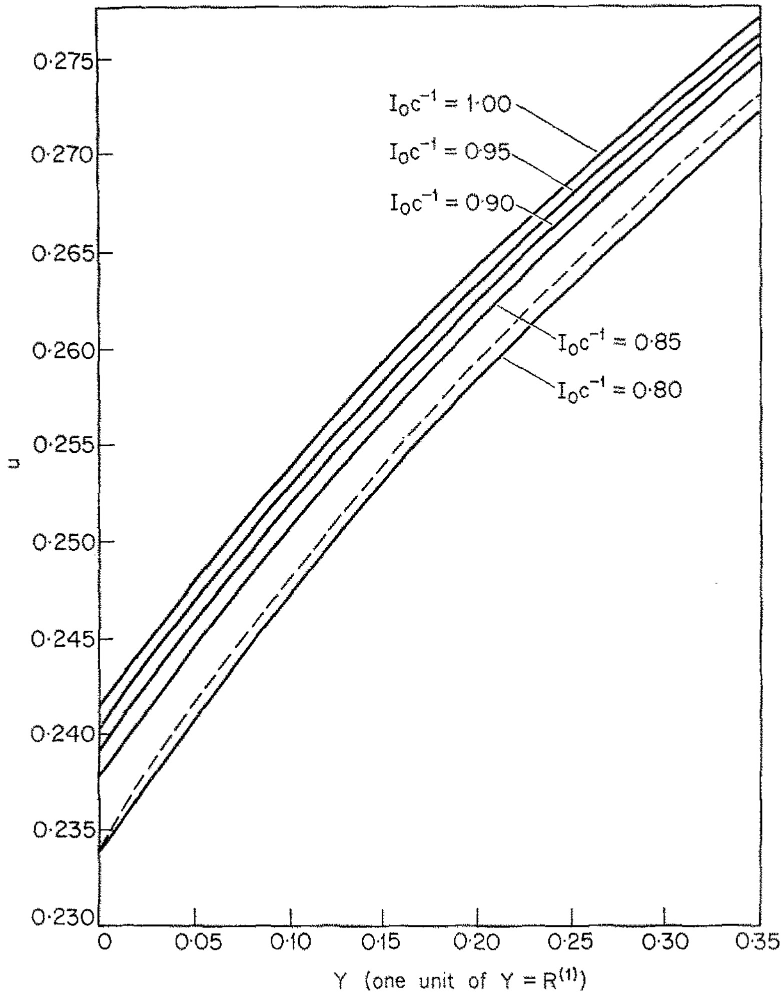

Figure 6.—Solution of equations (2.3.9) and (2.3.10) for u at the particular instant $t = t_0 = 0{\cdot}4$ for different values of the parameter $I_0 c^{-1}$ and the corresponding approximate solution for u (shown in dashed line) in the range of $Y = [0, 0{\cdot}35]$.

a solution of homogeneous differential system with constant I_0, Ψ_0, whereas in the Riemannian space we have non-homogeneous equations with variable I and Ψ.

4. *Numerical Example*†

Assume a hypothetical celestial body with the magnitude of the gravitational potential on its surface equal to $\psi^{(1)} = 8{\cdot}5 \times 10^4 \psi_s$, where ψ_s is the gravitational potential of the sun on its surface, $\psi_s = 7{\cdot}34 \times 10^4$ msec^{-2}. The velocity of the propagation of light is $c = 1{\cdot}86272 \times 10^5$ msec^{-1}. The value of $2c^{-2}\psi^{(1)}$ equals to 0·36, $c^{-1}I_0^{(1)} = 0{\cdot}8$. The gravitational potential is calculated at points $R^{(n)} = nR^{(1)}$, $R^{(1)}$ = radius of the celestial body. The quantities $c^{-2}\psi$, and $c^{-1}I_0$ as functions of $nR^{(1)}$ are shown on Fig. 1.

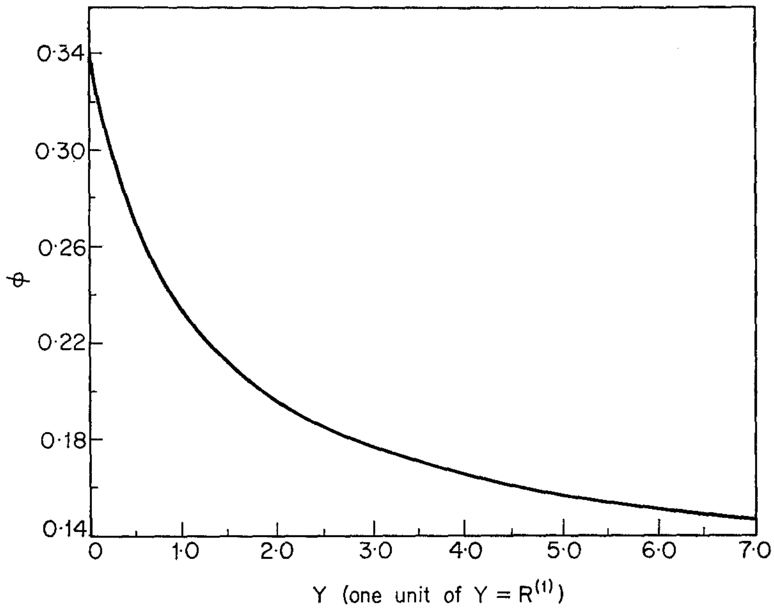

Figure 7.—Approximate solution of equations (2.3.9) and (2.3.10) for $\phi = \phi(Y, 0{\cdot}4)$. Y at $t = 0{\cdot}4$ due to the variations of the parameter $I_0 c^{-1}$ is so small that it cannot be shown clearly on this diagram.

Let us assume a hypothetical gas consisting only of electrons at a high temperature moving in the gravitational field of the celestial body: $p(\rho^0)^{-1} = \mathscr{R} m_w^{-1} \theta$, where $\mathscr{R}$ = universal constant = 1545·33 (ft lbf mole^{-1} °R^{-1}), m_w = molecular weight of the gas = $(1836)^{-1}$ lbm, θ temperature in degrees Rankine, p = pressure (lbf ft^{-2}), ρ^0 = density (lbm ft^{-3}). We solve

† The example was proposed by the author, was set up by Mr. A. El-Ariny, and calculated by Mr. Floyd E. LeCureux (both graduate students in the College of Engineering, Michigan State University) on the MSU Control Data 3600 Computer. The diagrams were plotted by Mr. El-Ariny.

numerically (2.3.9) and (2.3.10) in the Y-space-time with the initial conditions:

$$u_0 = u(0,0) = 0{\cdot}2; \qquad \rho_0{}^0 = \rho^0(0,0) = 10^{-15} \text{ lbm ft}^{-3};$$

$$\theta_0 = \theta(0,0) = 1{\cdot}225(10^9)\ {}^\circ\text{R}, \qquad u = u(Y,t) \tag{4.1}$$

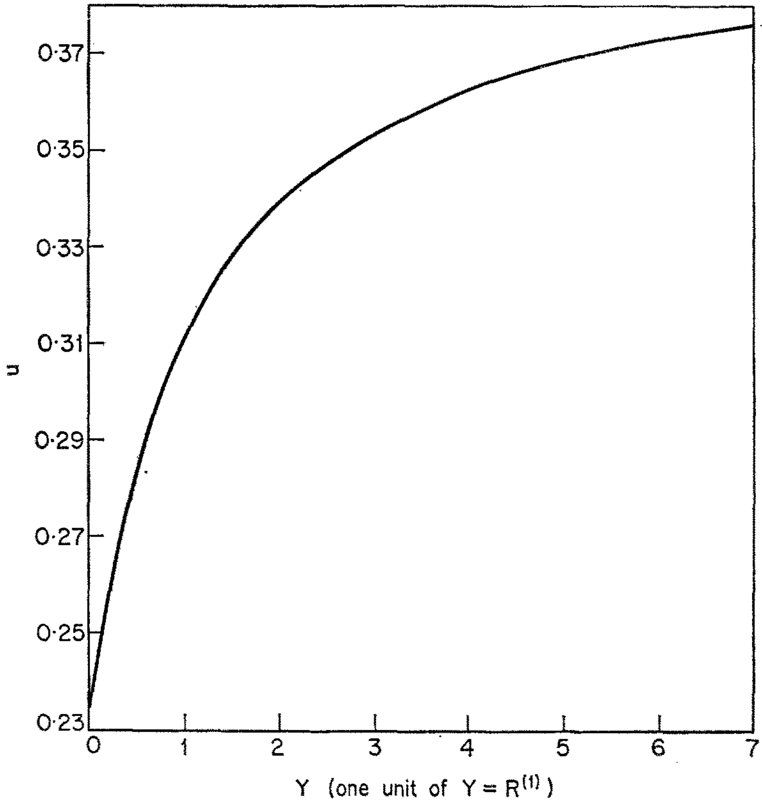

Figure 8.—Approximate solution of equations (2.3.9) and (2.3.10) for $u = u(Y, 0{\cdot}4)$. The variation of the curve u as a function of Y at $t = 0{\cdot}4$ due to the variations of the parameter $I_0 c^{-1}$ is so small that it cannot be shown clearly on this diagram.

with the origin of the Y-coordinate located on the surface of the celestial body and:

$$u(Y,0) = u_0[1 + Y(1+Y)^{-1}]; \qquad \varphi(Y,0) = \tfrac{2}{3}\varphi_0[\tfrac{3}{2} - Y(1+Y)^{-1}] \tag{4.2}$$

We easily calculate the value of $c^{-2}p_0(\rho_0{}^0)^{-1} = 0{\cdot}115$, and from (2.2.2) and $\varepsilon = (\gamma - 1)^{-1}p(\rho^0)^{-1}$ we get $\gamma_0 \leqslant 1{\cdot}614$. It is assumed throughout this section that $\gamma_0 = 1{\cdot}614$. Since the gas is assumed to be an isentropic one, we get:

$$p = C(\rho^0)^\gamma \qquad \text{and} \qquad c^{-2}C = 1{\cdot}8676 \times 10^8, \qquad \gamma = \gamma_0 \tag{4.3}$$

The density is normalized with respect to $\rho_0{}^0$, i.e., $\rho^{0*} = \rho^0(\rho_0{}^0)^{-1}$. From Table 1 we calculate α as a function of ρ^{0*} (Fig. 2). From $(2.3.8)_2$ using α from Table 1 with $\varphi = 0$ for $\rho^{0*} = 0$ (see Chernikov, 1962), we may plot

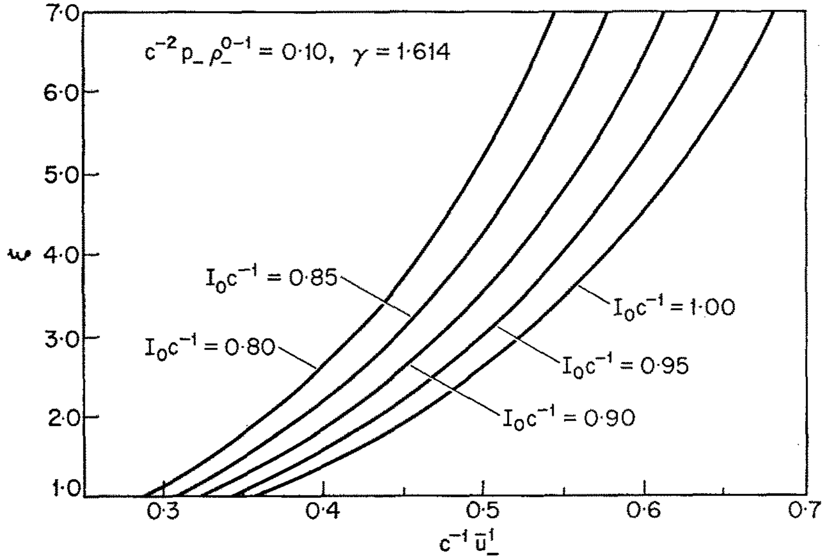

Figure 9.—The pressure ratio $\xi = p_+ p_-^{-1}$ versus the dimensionless shock velocity $c^{-1}\bar{u}_-^1$ for different values of the parameter $c^{-1}I_0$ when $c^{-2}p_-\rho^{0-1} = 0{\cdot}10$ and $\gamma = 1{\cdot}614$.

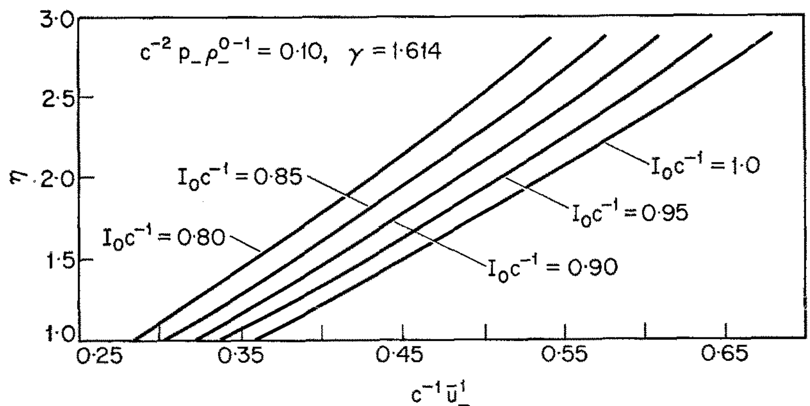

Figure 10.—The density ratio $\eta = \rho_+{}^0 \rho_-^{0-1}$ versus $c^{-1}\bar{u}_-^1$ for different values of the parameter $c^{-1}I_0$ when $c^{-2}p_-\rho_-^{0-1} = 0{\cdot}10$ and $\gamma = 1{\cdot}614$.

$\varphi = \varphi(\rho^{0*})$ (Fig. 3). The quantity $\varphi_0 = \varphi(0,0)$ in (4.1) is determined from Fig. 3 for $\rho_0^{0*} = 1$ ($\varphi_0 = 0{\cdot}354466$). From Figs. 2 and 3 we get $\alpha = \alpha(\varphi)$ (Fig. 4). Figures 5 and 6 represent solutions of (2.3.9) and (2.3.10), respectively, for different constant parameters $I_{(n)}^0$ taken at a particular instant,

$t = t_0$. Only small portions of these curves are represented on Figs. 5 and 6.

Using Fig. 1 we determine the positions $Y = Y^{(n)} = R^{(n)}$ at which the values of the parameter $I_0 = I_0^{(n)}$ are chosen. In Figs. 5 and 6 (extended up to $Y = 7{\cdot}0$) vertical lines are traced for each $Y^{(n)}$ so obtained. Next, there are determined the points of intersections of these vertical lines with the corresponding curves drawn for the corresponding values of the parameter $I_0 = I_0^{(n)}$. The curves, passing through these points, Figs. 7 and 8, furnish the first approximate solutions of (2.3.9) and (2.3.10), respectively, for the case of a piece-wise variable reference velocity, I_0. Small portions of these curves are marked on Figs. 5 and 6, as dashed lines.

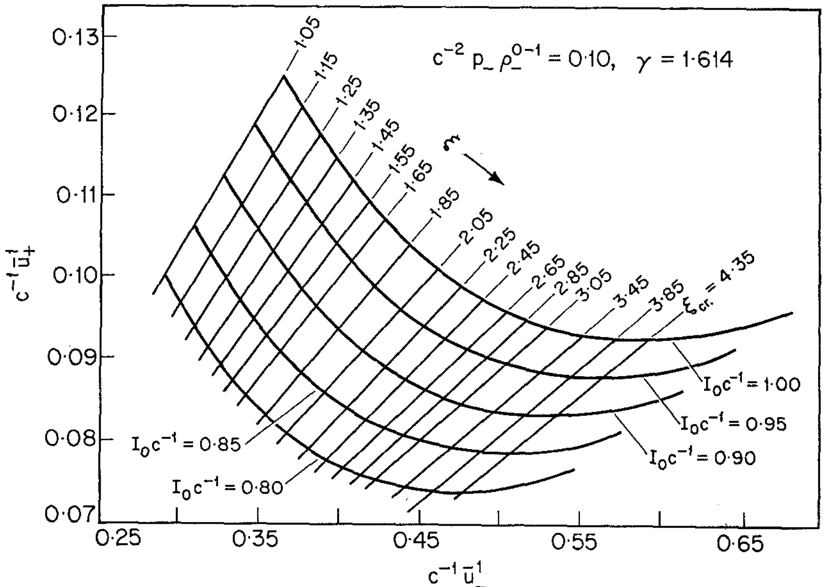

Figure 11.—The dimensionless velocity $c^{-1}\bar{u}_+{}^1$ on the right side of the shock versus the dimensionless velocity $c^{-1}\bar{u}_-{}^1$ on the left side of the shock for different values of the parameter $c^{-1}I_0$ when $c^{-2}p_-\rho_-^{0-1} = 0{\cdot}10$ and $\gamma = 1{\cdot}614$. Constant pressure ratio lines are also shown.

The numerical calculations of (2.3.15), (2.3.20) and (2.3.22) with (2.3.18) and (2.3.19) refer to a stationary shock normal to Y^1-axis. We may specify $(\rho_-{}^0)$ and (p_-) on the right-hand side of the shock, and (p_+) or ξ (the strength of the shock) on the left-hand side (Dufay, 1957). The variables $(\rho_+{}^0)$ or η, u_-, u_+, are calculated. The quantity $(p_-(\rho_-{}^0)^{-1})$ on the left-hand side of the shock is kept constant. Figures 9, 10 and 11, show the relations (ξ versus $\bar{u}_-{}^1$), (η versus $\bar{u}_-{}^1$), and ($\bar{u}_+{}^1$ versus $\bar{u}_-{}^1$), respectively, for various constant I_0. The linear relation $\bar{u}_\pm{}^1$ versus $u_\pm$ for different I_0 is shown on Fig. 12. All curves are similar, but no attempt was made to find the center of similarity.

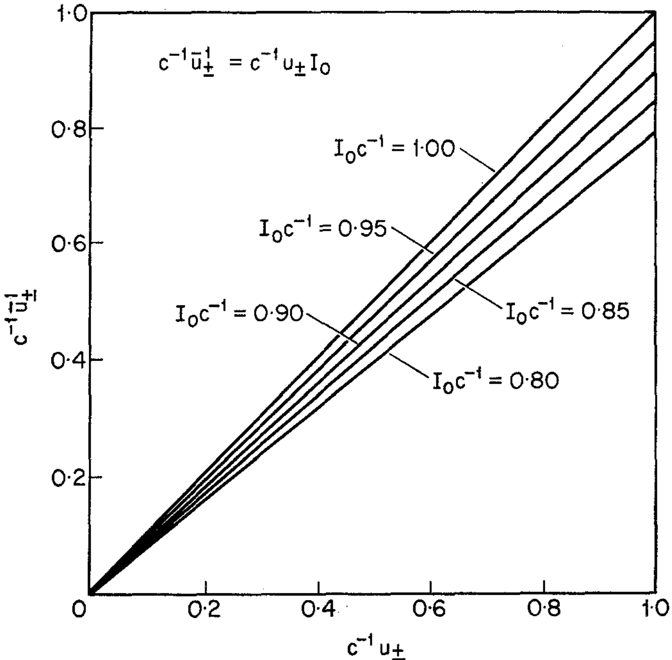

Figure 12.—$c^{-1}\bar{u}_{\pm}{}^1$ versus $c^{-1}u_{\pm}$ for different values of the parameter $c^{-1}I_0$.

Figures 9 and 10 seem to indicate that the shock parameters ξ and η increase as ψ increases or I_0 decreases with $\bar{u}_-{}^1$ constant. For fixed ξ and η, $\bar{u}_-{}^1$ increases as ψ decreases or I_0 increases. Figure 11 seems to demonstrate that $\bar{u}_+{}^1$ increases as ψ decreases or I_0 increases for $\bar{u}_+{}^1$ constant, or vice versa. There appears in Fig. 11 ($\xi = \xi_{cr}$) a minimum the meaning of which has never been discussed.

References

Bateman, H. (1910). The Transformation of the Electrodynamical Equations. *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 8, 223–264.

Chernikov, N. A. (1962). Flux Vector and Mass Tensor of a Relativistic Ideal Gas. *Soviet Physics: Doklady*, Vol. 7, No. 5, 414–416.

Dufay, J. (1957). *Galactic Nebulae and Interstellar Matter*. Translated from the Russian by Pomerans. New York Philosophical Library.

Einstein, A. (1905). Zur Elektrodynamik Bewegter Koerper. *Annalen der Physik*, Vierte Folge, Band 17, 891–921.

Einstein, A. (1907). Ueber das Relativitaetsprinzip und die aus demselbem gezogenen Folgerungen. *Jahrbuch der Radioaktivitat und Elektronik*, **4**, 411–457. Ibid., **5**, 98.

Fock, V. (1959). *The Theory of Space and Gravitation*. Pergamon Press, London. (Translated from the Russian, 1955.)

Møller, C. (1952). *The Theory of Relativity*. Clarendon Press, Oxford.

Pauli, W. (1958). *Theory of Relativity*. Pergamon Press, London.

Rosen, N. (1952). Special Theories of Relativity. *American Journal of Physics*, Vol. 20, No. 3 (March 1952), 161–164.

Synge, J. L. (1956). *Relativity: The Special Theory*. North-Holland Publishing Co. In USA: Interscience Publishers Inc., New York.

Taub, A. (1948). Relativistic Rankine-Hugoniot Equations. *Physical Review*, Vol. 74, No. 3 (August 1, 1948), 328–334.

Trautmann, A., Pirani, F. A. E. and Bondi, H. (1964). *Lectures on General Relativity. Brandeis Institute in Theoretical Physics*, Vol. 1. Prentice-Hall, Inc., New York.

Whitehead, A. N. (1922). *The Principle of Relativity with Applications to Physical Sciences*. Cambridge, The University Press.